EMBEDDED MINIMAL DISKS WITH PRESCRIBED CURVATURE BLOWUP

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Abstract. We construct a sequence of compact embedded minimal disks in a ball in $\mathbb{R}^3$, whose boundaries lie in the boundary of the ball, such that the curvature blows up only at a prescribed discrete (and hence, finite) set of points on the x_3 − axis. This extends a result of Colding and Minicozzi, who constructed a sequence for which the curvature blows up only at the center of the ball, and is a partial affirmative answer to the larger question of the existence of a sequence for which the curvature blows up precisely on a prescribed closed set on the x_3 − axis.

In [1], T.H. Colding and W.P. Minicozzi II constructed a sequence of compact embedded minimal disks in a ball in $\mathbb{R}^3$, with boundaries lying in the boundary of the ball, such that the curvature blows up only at the center. This result raises the following question.

Question 1. *Does there exist a sequence of compact embedded minimal disks in a ball in $\mathbb{R}^3$, whose boundaries lie in the boundary of the ball, such that the curvature blows up precisely on a prescribed closed set on the $x_3 -$ axis?*

Beyond that, it is interesting to consider which curves can arise as the singular set for curvature of a sequence of embedded minimal disks. W. Meeks and M. Weber have constructed examples (see [3]) in which the singular set is a circle.

By scaling, it suffices to consider embedded minimal disks in the unit ball. Our main result says that the answer to Question 1 is affirmative in the case where the closed set is a discrete (and hence, finite) set of points.

Theorem 2. *Given n points $(0,0,b_j) \subset B_1$, $b_1 < \ldots < b_n$, there is a sequence of compact embedded minimal disks $0 \in \Sigma_i \subset B_1 \subset \mathbb{R}^3$ with $\partial\Sigma_i \subset \partial B_1$ and containing the vertical segment $\{(0,0,t) : |t| < 1\} \subset \Sigma_i$, and such that the following hold:*

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(i) $\lim_{i\to\infty} |A_{\Sigma_i}|^2(0,0,b_j) = \infty$, $j = 1,\dots,n$

(ii) $\sup_i \sup_{\Sigma_i \setminus \cup_j B_{\delta_j}(0,0,b_j)} |A_{\Sigma_i}|^2 < \infty$ *for all* $\delta_j > 0$, $j = 1,\dots,n$

(iii) $\Sigma_i \setminus \{x_3 - axis\} = \Sigma_{1,i} \cup \Sigma_{2,i}$ *for multi-valued graphs* $\Sigma_{1,i}$ *and* $\Sigma_{2,i}$

(iv) $\Sigma_i \setminus \cup_j \{x_3 = b_j\}$ *converges to* $n+1$ *embedded minimal disks* Σ^k, $k = 1,\dots,n+1$, *satisfying the following:*

(a) $\Sigma^k \subset \{x_3 < b_k\}$ *for* $k = 1,\dots,n$, *and* $\Sigma^{n+1} \subset \{x_3 > b_n\}$

(b) $\overline{\Sigma^1}\setminus\Sigma^1 = B_1 \cap \{x_3 = b_1\}$, $\overline{\Sigma^{n+1}}\setminus\Sigma^{n+1} = B_1 \cap \{x_3 = b_n\}$, *and for* $k = 2,\dots,n$, $\overline{\Sigma^k}\setminus\Sigma^k = B_1 \cap (\{x_3 = b_{k-1}\} \cup \{x_3 = b_k\})$

(c) $\Sigma^1 \setminus \{x_3 - axis\} = \Sigma^1_1 \cup \Sigma^1_2$ *for multi-valued graphs* Σ^1_1 *and* Σ^1_2 *each of which spirals into* $\{x_3 = b_1\}$. $\Sigma^{n+1}\setminus\{x_3 - axis\} = \Sigma^{n+1}_1 \cup \Sigma^{n+1}_2$ *for multi-valued graphs* Σ^{n+1}_1 *and* Σ^{n+1}_2 *each of which spirals into* $\{x_3 = b_n\}$. *For* $k = 2,\dots,n$, $\Sigma^k \setminus \{x_3 - axis\} = \Sigma^k_1 \cup \Sigma^k_2$ *for multi-valued graphs* Σ^k_1 *and* Σ^k_2 *each of which spirals into* $\{x_3 = b_{k-1}\}$ *and* $\{x_3 = b_k\}$.

Question 1 remains open for closed sets in general; for example, closed intervals or Cantor-type sets. We conjecture that the answer is affirmative in general. Let us briefly discuss how one might show this. One idea is to note that, given any closed set, there exists a countable dense subset. One would want to construct a sequence of compact embedded minimal disks whose curvature blows up on the countable dense subset. By Theorem 2, such a sequence exists for any set of n points, for any fixed finite n; one would then want to let $n \to \infty$ and use a diagonal argument to obtain a sequence whose curvature blows up on the countable set of points in the dense subset. As a result of [2, Lemma I.1.4], the set of points on which the curvature blows up must be closed. Hence, the curvature would blow up on the closure of the countable dense subset, which is precisely our prescribed closed set.

The key to extending Theorem 2 from finitely many to countably many points would be to show that all of the intermediate results we use in this paper to prove Theorem 2 hold uniformly in n. As we prove these intermediate results, most of them will be easily seen to hold uniformly in n. However, it is not clear whether or not part (iii) in Lemma 5 is uniform; it appears that the number r_0 which we obtain depends on n, and approaches 0 as n tends to infinity.

We now return to the issue at hand: the finitely many points case. Theorem 2 says the following. Given n points on the x_3-axis, $(0,0,b_j)$ for $j = 1,\dots,n$, with $b_1 < \dots < b_n$, we construct a sequence of disks $\Sigma_i \subset B_1 = B_1(0) \subset \mathbb{R}^3$ where the curvatures blow up only at the prescribed n points, and $\Sigma_i \setminus \{x_3 - \text{axis}\}$ consists of two multi-valued graphs for each i. The sequence $\Sigma_i \setminus \cup_j \{x_3 = b_j\}$ converges to $n+1$ embedded minimal disks Σ^k, which sit between and spiral into the

appropriate planes $\{x_3 = b_j\}$. The result of Colding and Minicozzi in [1] is just Theorem 2 with $n = 1$ and $b_1 = 0$.

For the reader's convenience, we will structure this paper similarly to [1]. In particular, we provide some of the brief background on the Weierstrass representation which Colding and Minicozzi also outlined.

Let $\Omega \subset \mathbb{C}$ be a domain. The Weierstrass representation is as follows (see, for example, [4]). Given any meromorphic function g on Ω and any holomorphic one-form ϕ on Ω, we obtain a (branched) conformal minimal immersion $F : \Omega \to \mathbb{R}^3$, where

$$F(z) = \operatorname{Re} \int_{\zeta \in \gamma_{z_0,z}} \left(\frac{1}{2}(g^{-1}(\zeta) - g(\zeta)), \frac{i}{2}(g^{-1}(\zeta) + g(\zeta)), 1 \right) \phi(\zeta). \tag{1}$$

Here, we are integrating along a path $\gamma_{z_0,z}$ from a fixed base point z_0 to z. The choice of z_0 changes F by adding a constant. We will assume that $F(z)$ is independent of the choice of path, which is the case, for example, when g has no zeros or poles and Ω is simply connected (and this will be the case for our choices of g and Ω).

The unit normal $\mathbf{n}$ and Gauss curvature K of the resulting minimal surface are given by (see [4, Sec. 8,9])

$$\mathbf{n} = (2 \operatorname{Re} g, 2 \operatorname{Im} g, |g|^2 - 1)/(|g|^2 + 1), \tag{2}$$

$$K = - \left[\frac{4|\partial_z g||g|}{|\phi|(1 + |g|^2)^2} \right]^2 . \tag{3}$$

The one-form ϕ is called the *height differential*, and by equation (2), g is the composition of the Gauss map followed by stereographic projection.

We will assume that ϕ does not vanish and g has no zeros or poles; this implies that F is an immersion, i.e., $dF \neq 0$. One of the standard examples of this, which has the added benefit of being an ∞-valued graph, and hence interesting for our purposes, is the helicoid, whose Weierstrass data are

$$g(z) = e^{iz},\ \phi(z) = dz,\ \Omega = \mathbb{C}. \tag{4}$$

This motivates the following. If we want to construct multi-valued minimal graphs, perhaps we should consider Weierstrass data of the form

$$g(z) = e^{ih(z)} = e^{i(u(z)+iv(z))},\ \phi(z) = dz,$$

for an appropriate choice of Ω, where $h(z)$ is a holomorphic function. The next lemma gives us the differential of F in this case.

Lemma 3. *If F is given by equation (1) with $g(z) = e^{i(u(z)+iv(z))}$ and $\phi(z) = dz$, then*

$$\partial_x F = (\sinh v\, \cos u, \sinh v\, \sin u, 1), \tag{5}$$

$$\partial_y F = (\cosh v\, \sin u, -\cosh v\, \cos u, 0). \tag{6}$$

In particular, for the proof of Theorem 2, we will construct our multi-valued minimal graphs in this way, with our choices of function $h_a(z)$ and domain Ω_a varying for each element of the sequence. That is, we will construct a one-parameter family of minimal immersions F_a, $a \in (0, 1/2)$, with Weierstrass data $g = e^{ih_a}$ (where $h_a = u_a + iv_a$), $\phi = dz$, and domains Ω_a which we will specify shortly. We will prove that this family of immersions is compact in Lemma 4, and that the immersions $F_a : \Omega_a \to \mathbb{R}^3$ are embeddings in Lemma 5.

For each $0 < a < 1/2$, let

$$h_a(z) = \sum_{j=1}^{n} \frac{1}{2^{j-1}a} \arctan\left(\frac{z - b_j}{a}\right) \text{ on } \Omega_a = \cup_{j=1}^{n} \Omega_{a,j}, \text{ where} \tag{7}$$

$$\begin{aligned}
\Omega_{a,1} &= \left\{(x,y) : -\frac{1}{2} \le x \le \frac{b_2 - b_1}{2},\ |y| \le \frac{[(x-b_1)^2 + a^2]^{3/4}}{2}\right\} \\
\Omega_{a,j} &= \left\{(x,y) : \frac{b_j - b_{j-1}}{2} \le x \le \frac{b_{j+1} - b_j}{2},\ |y| \le \frac{[(x-b_j)^2 + a^2]^{3/4}}{2}\right\}, \\
&\quad j = 2, \dots, n-1 \\
\Omega_{a,n} &= \left\{(x,y) : \frac{b_n - b_{n-1}}{2} \le x \le \frac{1}{2},\ |y| \le \frac{[(x-b_n)^2 + a^2]^{3/4}}{2}\right\}.
\end{aligned}$$

To get an idea of what Ω_a looks like, note that the $\Omega_{a,j}$ are defined similarly to the domain called Ω_a by Colding and Minicozzi (see [1, Figure 4]), only centered at b_j instead of at 0. When $a \to 0$, the domain pinches off at the n points b_j, just as Colding and Minicozzi's domain pinches off at 0 (see [1, Figure 5]).

Note that h_a is well-defined, since Ω_a is simply connected and $b_j \pm ia \notin \Omega_a$ for $j = 1, \dots, n$. By direct computation, we see that

$$\begin{aligned}
\partial_z h_a(z) &= \sum_{j=1}^{n} \frac{1}{2^{j-1}} \frac{1}{(z-b_j)^2 + a^2} \\
&= \sum_{j=1}^{n} \frac{1}{2^{j-1}} \frac{(x-b_j)^2 + a^2 - y^2 - 2i(x-b_j)y}{[(x-b_j)^2 + a^2 - y^2]^2 + 4(x-b_j)^2 y^2}.
\end{aligned} \tag{8}$$

By the Cauchy-Riemann equations, we get

$$\partial_z h_a = \partial_x u_a - i\partial_y u_a = \partial_y v_a + i\partial_x v_a. \tag{9}$$

Also, the curvature is given by (see equation (3))

$$\begin{aligned} K_a(z) &= \frac{-|\partial_z h_a|^2}{\cosh^4 v_a} \\ &= -\frac{|\sum_{j=1}^n 2^{1-j}((z-b_j)^2+a^2)^{-1}|^2}{\cosh^4(\mathrm{Im}(\sum_{j=1}^n \arctan((z-b_j)/a)/2^{j-1}a))}. \end{aligned} \tag{10}$$

Note that $\lim_{a\to 0}|K_a(z)| = \infty$ for $z = b_j$, $j = 1, \dots, n$.

Let $F_a : \Omega_a \to \mathbb{R}^3$ be from equation (1) with $g = e^{ih_a}$, $\phi = dz$, and $z_0 = 0$. Let $\Omega_0 = \cap_a \Omega_a \setminus \{b_1, \dots, b_n\}$. The family of functions h_a is not compact, since $\lim_{a\to 0}|h_a|(z) = \infty$ for $z \in \Omega_0$. However, as the following lemma shows, the family of immersions F_a is compact.

Lemma 4. *If $a_k \to 0$, there exists a subsequence, which we also call a_k, such that F_{a_k} converges uniformly in C^2 on compact subsets of Ω_0.*

Proof. Similar to the proof of [1, Lemma 2], with

$$-\sum_{j=1}^n \frac{1}{2^{j-1}} \frac{1}{z-b_j}$$

in place of -1/z. $\square$

In the next lemma, we show that the immersions $F_a : \Omega_a \to \mathbb{R}^3$ are in fact embeddings. This will follow from parts (i) and (ii) of the lemma. Part (i) says that the slice $\{x_3 = t\} \cap F_a(\Omega_a)$ is the image of the segment $\{x = t\}$ in the plane; that is, as x varies and y stays fixed, there is no self-intersection. In part (ii), we show that, in each slice $\{x_3 = t\} \cap F_a(\Omega_a)$, the image $F_a(\{x = t\} \cap \Omega_a)$ is a graph over some line segment in the slice; that is, as y varies and x stays fixed, there is no self-intersection.

Lemma 5. *For all $a > 0$, the immersions $F_a : \Omega_a \to \mathbb{R}^3$ satisfy*

(i) $x_3(F_a(x, y)) = x$
(ii) *For each fixed x, $F_a(x, \cdot)$ is a graph in the plane $\{x_3 = x\}$.*

(iii) *There exists $r_0 > 0$ such that, for all a,*

$$
\begin{aligned}
\left| F_a\left(x, \pm\frac{[(x-b_1)^2+a^2]^{3/4}}{2}\right) - F_a(x,0)\right| &> r_0,\ -\frac{1}{2} \le x \le \frac{b_2-b_1}{2} \\
\left| F_a\left(x, \pm\frac{[(x-b_j)^2+a^2]^{3/4}}{2}\right) - F_a(x,0)\right| &> r_0,\ \frac{b_j-b_{j-1}}{2} \le x \le \frac{b_{j+1}-b_j}{2}, \\
&\quad\ j = 2,\dots,n-1 \\
\left| F_a\left(x, \pm\frac{[(x-b_n)^2+a^2]^{3/4}}{2}\right) - F_a(x,0)\right| &> r_0,\ \frac{b_n-b_{n-1}}{2} \le x \le \frac{1}{2}.
\end{aligned}
$$

Proof. (i) is immediate by the definition of F_a, since $z_0 = 0$ and $\phi = dz$.

To prove (ii), first note that, by equations (8) and (9), we have

$$|\partial_y u_a(x,y)| \le \sum_{j=1}^{n} \frac{1}{2^{j-1}} \frac{2|x-b_j||y|}{[(x-b_j)^2+a^2-y^2]^2+4(x-b_j)^2y^2}.$$

Fix k, $1 \le k \le n$. On $\Omega_{a,k}$ (where $(x-b_k)^2 = \min_j (x-b_j)^2$), we have, for all $j = 1,\dots,n$,

$$
\begin{aligned}
[(x-b_j)^2+a^2-y^2]^2+4(x-b_j)^2y^2 &\ge [(x-b_j)^2+a^2-y^2]^2 \\
&\ge \left[(x-b_j)^2+a^2-\frac{(x-b_k)^2+a^2}{4}\right]^2 \\
&\ge \left[(x-b_j)^2+a^2-\frac{(x-b_j)^2+a^2}{4}\right]^2 \\
&= \frac{9}{16}[(x-b_j)^2+a^2]^2.
\end{aligned}
$$

Therefore, we have

$$|\partial_y u_a(x,y)| \le 4\sum_{j=1}^{n} \frac{1}{2^{j-1}} \frac{|x-b_j||y|}{[(x-b_j)^2+a^2]^2}. \tag{11}$$

Set $y_{x,a,k} = \frac{[(x-b_k)^2+a^2]^{3/4}}{2}$. Integrating (11) gives

$$
\begin{aligned}
\max_{|y|\leq y_{x,a,k}} |u_a(x,y) - u_a(x,0)| &\leq \max_{|y|\leq y_{x,a,k}} \left| \int_0^y \partial_y u_a(x,t)\,dt \right| \\
&\leq \int_0^{y_{x,a,k}} 4 \sum_{j=1}^n \frac{1}{2^{j-1}} \frac{|x-b_j|t}{[(x-b_j)^2+a^2]^2}\,dt \\
&= 2 \sum_{j=1}^n \frac{1}{2^{j-1}} \frac{|x-b_j|t^2}{[(x-b_j)^2+a^2]^2} \Big|_0^{y_{x,a,k}} \\
&= 2 \sum_{j=1}^n \frac{1}{2^{j-1}} \frac{|x-b_j|}{[(x-b_j)^2+a^2]^2} \frac{[(x-b_k)^2+a^2]^{3/2}}{4} \\
&\leq \frac{1}{2} \sum_{j=1}^n \frac{1}{2^{j-1}} \frac{|x-b_j|}{[(x-b_j)^2+a^2]^2} [(x-b_j)^2+a^2]^{3/2} \\
&= \frac{1}{2} \sum_{j=1}^n \frac{1}{2^{j-1}} \frac{|x-b_j|}{[(x-b_j)^2+a^2]^{1/2}} \\
&\leq \frac{1}{2} \sum_{j=1}^n \frac{1}{2^{j-1}} \\
&< \frac{1}{2} \sum_{j=1}^\infty \frac{1}{2^{j-1}} \\
&= 1. \qquad (12)
\end{aligned}
$$

Set $\gamma_{x,a}(y) = F_a(x,y)$. Since $v_a(x,0) = 0$ and $\cos(1) > 1/2$, combining (6) and (12), we get

$$
\begin{aligned}
\langle \gamma'_{x,a}(y), \gamma'_{x,a}(0) \rangle &= \cosh v_a(x,y) \cos(u_a(x,y) - u_a(x,0)) \\
&> \cosh v_a(x,y)/2, \qquad (13)
\end{aligned}
$$

where $\gamma'_{x,a}(y) = \partial_y F_a(x,y)$. By (13), the angle between $\gamma'_{x,a}(y)$ and $\gamma'_{x,a}(0)$ is always less than $\pi/2$, proving (ii) on $\Omega_{a,k}$, and hence on all of Ω_a since k was arbitrary.

To prove (iii), note that, by (8) and (9), we have

$$
\partial_y v_a(x,y) = \sum_{j=1}^n \frac{1}{2^{j-1}} \frac{(x-b_j)^2 + a^2 - y^2}{[(x-b_j)^2+a^2-y^2]^2 + 4(x-b_j)^2 y^2}.
$$

As before, fix k, $1 \le k \le n$, and look on $\Omega_{a,k}$. Then, for all $j = 1, \dots, n$,

$$\begin{aligned}
[(x-b_j)^2 + a^2 - y^2]^2 + 4(x-b_j)^2 y^2 &\le [(x-b_j)^2 + a^2 + y^2]^2 \\
&\le \left[(x-b_j)^2 + a^2 + \frac{(x-b_k)^2 + a^2}{4}\right]^2 \\
&\le \left[(x-b_j)^2 + a^2 + \frac{(x-b_j)^2 + a^2}{4}\right]^2 \\
&= \frac{25}{16}[(x-b_j)^2 + a^2]^2.
\end{aligned}$$

$$\begin{aligned}
(x-b_j)^2 + a^2 - y^2 &\ge (x-b_j)^2 + a^2 - \frac{(x-b_k)^2 + a^2}{4} \\
&\ge (x-b_j)^2 + a^2 - \frac{(x-b_j)^2 + a^2}{4} \\
&= \frac{3}{4}[(x-b_j)^2 + a^2].
\end{aligned}$$

So, we have

$$\begin{aligned}
\partial_y v_a(x,y) &\ge \frac{12}{25} \sum_{j=1}^{n} \frac{1}{2^{j-1}} \frac{1}{(x-b_j)^2 + a^2} \\
&> \frac{3}{8} \sum_{j=1}^{n} \frac{1}{2^{j-1}} \frac{1}{(x-b_j)^2 + a^2}. \qquad (14)
\end{aligned}$$

Let $y_{x,a,k} = \frac{[(x-b_k)^2 + a^2]^{3/4}}{2}$, as before. Since $v_a(x,0) = 0$, integrating (14) gives

$$\begin{aligned}
\min_{\frac{y_{x,a,k}}{2} \le |y| \le y_{x,a,k}} |v_a(x,y)| &= \min_{\frac{y_{x,a,k}}{2} \le |y| \le y_{x,a,k}} \left| \int_0^y \partial_y v_a(x,t)\, dt \right| \\
&> \frac{3}{8} \int_0^{\frac{y_{x,a,k}}{2}} \sum_{j=1}^{n} \frac{1}{2^{j-1}} \frac{1}{(x-b_j)^2 + a^2}\, dt \\
&\ge \frac{3}{8} \int_0^{\frac{y_{x,a,k}}{2}} \frac{1}{2^{k-1}} \frac{1}{(x-b_k)^2 + a^2}\, dt \\
&= \frac{3}{8} \frac{1}{2^{k-1}} \frac{1}{(x-b_k)^2 + a^2} \frac{[(x-b_k)^2 + a^2]^{3/4}}{4} \\
&= \frac{3}{32} \frac{1}{2^{k-1}} [(x-b_k)^2 + a^2]^{-1/4} \\
&> \frac{[(x-b_k)^2 + a^2]^{-1/4}}{11 \cdot 2^{n-1}}. \qquad (15)
\end{aligned}$$

Now, integrating (13) and using (15), we obtain
(16)

$$\langle \gamma_{x,a}(y_{x,a,k}) - \gamma_{x,a}(0), \gamma'_{x,a}(0)\rangle > \frac{[(x-b_k)^2+a^2]^{3/4}}{16}\, e^{[(x-b_k)^2+a^2]^{-1/4}/11\cdot 2^{n-1}}.$$

Since $\lim_{s\to 0} s^3\, e^{s^{-1}/11\cdot 2^{n-1}} = \infty$, (16) and its analog for $\gamma_{x,a}(-y_{x,a,k})$ give an $r_k > 0$ for which (iii) holds (with r_k in place of r_0) on $\Omega_{a,k}$. This proves (iii) on all of Ω_a, with $r_0 = \min_k r_k$. □

Corollary 6. *Let r_0 be given by part (iii) of Lemma 5. Then,*

(a) *F_a is an embedding.*
(b) *$F_a(t,0) = (0,0,t)$ for $|t| < 1/2$.*
(c) *$\{0 < x_1^2 + x_2^2 < r_0^2\} \cap F_a(\Omega_a) = \widetilde{\Sigma}_{1,a} \cup \widetilde{\Sigma}_{2,a}$ for multi-valued graphs $\widetilde{\Sigma}_{1,a}$, $\widetilde{\Sigma}_{2,a}$ over $D_{r_0}\setminus\{0\}$.*

Proof. Same as [1, Cor. 1]. □

Proof of Theorem 2. By scaling, it suffices to find a sequence $\Sigma_i \subset B_R$ for some $R > 0$. By Corollary 6, there exist minimal embeddings $F_a : \Omega_a \to \mathbb{R}^3$ with $F_a(t,0) = (0,0,t)$ for $|t| < 1/2$, so (iii) holds for any $R \le r_0$. Set $R = \min\{r_0/2, 1/4\}$, and $\Sigma_i = B_R \cap F_{a_i}(\Omega_{a_i})$, where the sequence a_i is to be determined.

For each $j = 1, \ldots, n$, by equation (10), we have $|K_a|(b_j) \to \infty$ as $a \to 0$, proving (i).

Also by (10), for each $j = 1, \ldots, n$ and all $\delta > 0$,

$$\sup_a \sup_{\{|x-b_j|\ge\delta\}\cap\Omega_a} |K_a| < \infty$$

for all $x \notin \{b_1, ..., b_n\}$. Combined with (iii) and Heinz's curvature estimate for minimal graphs (see, for example, [4, 11.7]), this proves (ii).

By Lemma 4, we can choose $a_i \to 0$ so that the F_{a_i} converge uniformly in C^2 on compact subsets to $F_0 : \Omega_0 \to \mathbb{R}^3$. So, by Lemma 5, we obtain (iv)(a) and the decomposition $\Sigma^k\setminus\{x_3 - \text{axis}\} = \Sigma_1^k \cup \Sigma_2^k$ for multi-valued graphs Σ_j^k, where $j = 1,2$ and $k = 1, \ldots, n+1$. To obtain (iv)(b) and the remainder of (iv)(c), we must show that each graph Σ_j^k is ∞-valued, as this would imply the spiraling which we seek. By (iii) and (6), the level sets $\{x_3 = x\} \cap \Sigma_j^k$ are graphs over the line in the direction

$$\lim_{a\to 0}(\sin u_a(x,0), -\cos u_a(x,0), 0).$$

Since, for all $j = 1, \ldots, n$ and all t sufficiently close to b_j,

$$\lim_{a\to 0} |u_a(t-b_j, 0) - u_a(2(t-b_j), 0)| = \frac{1}{2(t-b_j)},$$

we see that, for t sufficiently close to b_j, $\{t-b_j < |x_3| < 2(t-b_j)\} \cap \Sigma_j^k$ contains an embedded N_t-valued graph, where $N_t \approx 1/4\pi(t-b_j) \to \infty$ as $t \to b_j$. This proves that each Σ_j^k spirals the way we claim, completing the proof of (iv). $\square$

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